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LETTER TO THE EDITOR

On exact solutions for damped anharmonic oscillators

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Abstract. For arbitrary functions f_1, f_2 and f_3 the anharmonic oscillator $\ddot{x}+f_1(t)\dot{x}+f_2(t)x+f_3(t)x^3=0$ cannot be solved in closed form (i.e. the general solution cannot be expressed as elliptic functions). We apply the Painlevé test to obtain the constraint on the functions f_1, f_2 and f_3 for which the equation passes the test. The constraint on f_1, f_2 and f_3 (i.e. the differential equation which f_1, f_2 and f_3 obey) is discussed and solutions are given.

For non-linear ordinary and partial differential equations the general solution usually cannot be given explicitly. It is desirable to have an approach to find out whether a given non-linear differential equation can be explicitly solved. We investigate the non-linear anharmonic oscillators

$$\ddot{x}+f_1(t)\dot{x}+f_2(t)x+f_3(t)x^3=0 \tag{1}$$

where f_1, f_2 and f_3 are smooth functions and $\dot{x}\equiv dx/dt$. We assume that $f_3\neq 0$. For arbitrary functions f_1, f_2 and f_3 the non-linear equation (1) cannot be explictly solved. We would like to find the constraint on f_1, f_2 and f_3 such that (1) can be solved.

Before we study the general case (1) we give a brief review of the special case

$$\ddot{x}+c_1\dot{x}+c_2x+x^3=0 \tag{2}$$

where c_1 and c_2 are constants. The equation of motion given by (2) can be derived from the explicitly time-dependent Lagrangian

$$L(t, x(t), \dot{x}(t))=\tfrac{1}{2}\,e^{c_1t}\dot{x}^2-e^{c_1t}V(x) \tag{3}$$

where

$$V(x)=\int_0^x f(s)\,ds. \tag{4}$$

The function f is given by $f(x)=c_2x+x^3$. The corresponding Hamiltonian is given by

$$H(t, x(t), p(t))=\tfrac{1}{2}\,e^{-c_1t}p^2+e^{c_1t}V(x). \tag{5}$$

We apply two different approaches to find the constraint on c_1 and c_2. In the first approach we transform (2) to an integrable equation. The transformations impose conditions on c_1 and c_2. The second approach is Kowalewski's asymptotic method (also called the Painlevé test or singular point analysis (compare [1] and references therein)). We show that both approaches lead to the same constraints on c_1 and c_2

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for (2). We assume that $c_1 \neq 0$. If $c_1 = 0$, then the general solution of (2) can be given in terms of Jacobi elliptic functions [2].

In the first approach we start from the ansatz

$$x(t) = u(\xi(t))v(t) \tag{6}$$

and determine the functions ξ and v so that u satisfies

$$\frac{d^2u}{d\xi^2} + au^3 = 0 \tag{7}$$

where a is a constant. Equation (7) is integrable in terms of Jacobi elliptic functions. Inserting ansatz (6) into (2) yields

$$u''\dot{\xi}^2 v + u'[\ddot{\xi}v + \dot{\xi}(2\dot{v} + c_1 v)] + u[\ddot{v} + c_1\dot{v} + c_2 v] + u^3v^3 = 0 \tag{8}$$

where $u' \equiv du/d\xi$. Consequently we require that

$$\ddot{v} + c_1\dot{v} + c_2 v = 0 \tag{9}$$

and

$$\ddot{\xi}v + \dot{\xi}(2\dot{v} + c_1 v) = 0. \tag{10}$$

If v and ξ satisfy (9) and (10) then (8) takes the form

$$u''\dot{\xi}^2 + u^3v^2 = 0 \tag{11}$$

where we have assumed that $v \neq 0$. In order to obtain (7) from (11) we have to require that

$$\dot{\xi}^2 = k^2v^2 \tag{12}$$

where k is a constant ($k \neq 0$) with $a = 1/k^2$. The solution to (9) is given by

$$v(t) = A\,e^{r_1 t} + B\,e^{r_2 t} \tag{13}$$

where A and B are the constants of integration and r_1 and r_2 are the two (in general different) roots of

$$r^2 + c_1 r + c_2 = 0. \tag{14}$$

Inserting (12) into (10) yields

$$3\dot{v} + c_1 v = 0 \tag{15}$$

where we have assumed that $v \neq 0$. Inserting the general solutions (13) of (9) into (15) we obtain

$$3(Ar_1\,e^{r_1 t} + Br_2\,e^{r_2 t}) + c_1(A\,e^{r_1 t} + B\,e^{r_2 t}) = 0. \tag{16}$$

Consequently

$$3r_1 + c_1 = 0. \tag{17a}$$

$$3r_2 + c_1 = 0. \tag{17b}$$

From (14) we obtain $r_{1,2} = -c_1/2 \pm (c_1^2/4 - c_2)^{1/2}$. Therefore

$$2c_1^2 - 9c_2 = 0. \tag{18}$$

Consequently, if condition (18) is satisfied the general solution of (2) can be found with the help of Jacobi elliptic functions.

Let us now apply Kowalewski's asymptotic method (compare [1] and references therein). Equation (2) is considered in the complex domain with c_1 and c_2 real. For the sake of simplicity we do not change the notation. Inserting the Laurent expansion

$$x(t)=\sum_{j=0}^{\infty} a_j(t-t_1)^{j-n} \tag{19}$$

where t_1 denotes the pole position, yields $n=1$ and $a_0^2=-2$. The expansion coefficients a_1, a_2 and a_3 are determined by

$$3a_1a_0=c_1 \tag{20a}$$

$$3a_2a_0=-c_2-3a_1^2 \tag{20b}$$

$$4a_3=c_1a_2+c_2a_1+a_1^3+6a_0a_1a_2. \tag{20c}$$

The expansion coefficient a_4 is arbitrary in expansion (19) if

$$c_1^2(2c_1^2-9c_2)=0. \tag{21}$$

This means $r=4$ is a so-called resonance (compare [1] and references therein). The solution $c_1=0$ is the trivial case. The condition $2c_1^2=9c_2$ is the same as we obtained in the first approach. To summarise: if $2c_1^2=9c_2$, then the general solution of (2) can be expressed in terms of Jacobi elliptic functions. For this case (i.e. $2c_1^2=9c_2$) we can find an explicitly time-dependent first integral, namely

$$I(t,x(t),\dot{x}(t))=\exp(\tfrac{4}{3}c_1t)\left[\left(\dot{x}+\frac{c_1x}{3}\right)^2+\tfrac{1}{2}x^4\right]. \tag{22}$$

Notice that the explicitly time-dependent Hamiltonian (5) is not a first integral.

To discuss the motion we consider (2) in the phase plane (x,y) with $\dot{x}\equiv y$. We have to distinguish between the cases $c_2\geqslant 0$ and $c_2<0$. We assume that $c_1>0$. For $c_2\geqslant 0$ we find one stable time-independent solution, namely $(0,0)$. Since $c_1>0$ all trajectories tend to this solution. For $c_2<0$ we obtain the following three time-independent solutions: $(0,0)$, $(+(-c_2)^{1/2},0)$, $(-(-c_2)^{1/2},0)$. Then the time-independent solution $(0,0)$ is unstable, whereas the other two are stable. Depending on the initial value $(x(t=0),y(t=0))$ the trajectories tend to one of these stable time-independent solutions.

Let us now consider (1). Here we apply Kowalewski's asymptotic method. Inserting the ansatz (19) into (1) we find at the resonance $r=4$ the condition

$$\begin{aligned}9f_3^{(4)}f_3^3&-54f_3^{(3)}f_3'f_3^2+18f_3^{(3)}f_3^3f_1-36(f_3'')^2f_3^2+192f_3''(f_3')^2f_3-78f_3''f_3'f_3^2f_1+36f_3''f_3^3f_2\\&+3f_3''f_3^3f_1^2-112(f_3')^4+64(f_3')^3f_3f_1+6(f_3')^2f_1'f_3^2-72(f_3')^2f_3^2f_2\\&+90f_3'f_2'f_3^3-27f_3'f_1''f_3^3-57f_3'f_1'f_3^3f_1+72f_3'f_3^3f_2f_1\\&-14f_3'f_3^3f_1^3-54f_2''f_3^4-90f_2'f_3^4f_1+18f_1^{(3)}f_3^4+54f_1''f_3^4f_1\\&+36(f_1')^2f_3^4-36f_1'f_3^4f_2+60f_1'f_3^4f_1^2-36f_3^4f_2f_1^2+8f_3^4f_1^4=0\end{aligned} \tag{23}$$

where $f'\equiv \mathrm{d}f/\mathrm{d}t$ and $f^{(4)}=f''''=\mathrm{d}^4f/\mathrm{d}t^4$. This means if this condition is satisfied then the expansion coefficient a_4 is arbitrary. Let us now discuss (23). It is obvious that we cannot give the general solution to (23). Thus we discuss special cases.

Case I. Let $f_1(t)=c_1$, $f_2(t)=c_2$, and $f_3(t)=c_3$, where c_1, c_2 and c_3 are constants ($c_3 \neq 0$). Then we obtain

$$c_3^4 c_1^2(2c_1^2 - 9c_2) = 0. \tag{24}$$

Thus we find condition (21) together with c_3 arbitrary.

Case II. Let $f_1 = 0$ and $f_3 = 1$. Then we find

$$f_2'' = 0. \tag{25}$$

The general solution is given by $f_2(t) = At + B$, where A and B are the constants of integration. Now equation (1) takes the form

$$\ddot{x} + (A + Bt)x + x^3 = 0. \tag{26}$$

This is a special case of the second Painlevé transcendent. The solution has no branch points, and are therefore uniform functions in t (see [1-3]).

Case III. Let $f_2(t) = 0$ and $f_3(t) = 1$. Then (23) takes the form

$$f_1''' + 3f_1''f_1 + 2(f_1')^2 + \tfrac{10}{3}f_1'(f_1)^2 + \tfrac{4}{9}f_1^4 = 0. \tag{27}$$

This equation admits the particular solutions

$$f_1(t) = 3/t \qquad f_1(t) = 3/2t. \tag{28}$$

Thus (27) admits more than one branch in the Painlevé analysis. Equation (27) does not pass the Painlevé test, because it admits non-integer resonances (rational resonances). However, the equation (27) passes the so-called weak Painlevé test (see [1] and references therein).

Case IV. A case where f_1, f_2 and f_3 are non-constant and satisfy (23) is given by

$$f_1(t) = 1/4t \qquad f_2(t) = 1/8t^2 \qquad f_3(t) = 1/32t^2. \tag{29}$$

Equation (1) together with the functions given by (29) arises in the Painlevé analysis of external driven anharmonic oscillators [4]. Equation (1) together with the functions given by (29) can be integrated exactly in terms of elliptic functions.

Case V. Equation (1) together with

$$f_1(t) = 1/4t \qquad f_2(t) = 1/8t^2 \qquad f_3(t) = -1/8t^2 \tag{30}$$

occurs in the Painlevé analysis of the Lorenz model [5]. The functions f_1, f_2 and f_3 satisfy (23). Then (1) together with the functions given by (30) can be solved in terms of elliptic functions as follows. Applying the transformation $x(t) = t^{1/4}g(t^{1/4})$ to (1) where f_1, f_2 and f_3 are given by (30) yields $\mathrm{d}^2g/\mathrm{d}s^2 = 2g^3$ with $s = t^{1/4}$.

Equation (1) also arises in this study of the non-linear partial differential equation

$$u_{\eta\xi} = u^3. \tag{31}$$

This equation admits the Lie symmetry vector field

$$-\eta\frac{\partial}{\partial\eta} - \xi\frac{\partial}{\partial\xi} + u\frac{\partial}{\partial u}. \tag{32}$$

This Lie symmetry vector field corresponds to the scale invariance of (31), i.e.

$$\eta \to \varepsilon^{-1}\eta \qquad \xi \to \varepsilon^{-1}\xi \qquad u \to \varepsilon u. \tag{33}$$

The symmetry vector field (32) leads to the similarity ansatz

$$u(\eta, \xi) = \frac{1}{\xi}f(s) \tag{34}$$

with the similarity variable $s = \eta/\xi$. Inserting this ansatz into (31) gives

$$\frac{d^2 f}{ds^2} + \frac{2}{s}\frac{df}{ds} + \frac{1}{s} f^3 = 0. \tag{35}$$

Consequently, we have $f_1(s) = 2/s$, $f_2(s) = 0$ and $f_3(s) = 1/s$. These functions satisfy (23) (with $s \to t$). Therefore, (35) passes the Painlevé test.

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